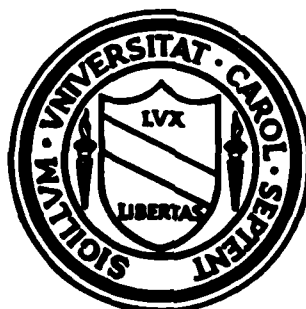


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Skewed Stable Variables and Processes

by

Clyde D. Hardin, Jr.

Technical Report #79

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Skewed Stable Variables and Processes

by

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Abstract: We consider here general (i.e. possibly skewed or asymmetric) stable distribution and processes. A decomposition result and a moment equality are given for these distributions. More importantly, we determine the form of all stable independent increments processes, construct a Wiener-type stochastic integral with respect to these processes, and prove a representation theorem for general stable processes analogous to (and in some sense including) the spectral representation theorem for symmetric stable processes.

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## Section 0. Introduction

One of the major themes of research in the area of stable processes has been the analysis of their similarities and differences with Gaussian processes, both of which there are many. A characteristic which stable distributions and processes may possess, yet which their Gaussian counterparts may never possess, is skewness (i.e. asymmetry in the distribution). Even though this is the case, most research in stable processes has dealt only with the symmetric case. While there is much left to be done in the symmetric case, it behooves us to investigate the asymmetric case as well, since it adds a new dimension to the theory and since the freedom to allow skewness enhances the potential for application involving these processes. This work is a modest step in that direction.

The outline of the paper is as follows. In section 1, we define the relevant terms and give some necessary (and some unnecessary) preliminary facts. In section 2, we characterize all stable independent increment processes, and develop a Wiener-type stochastic integral with respect to those with location parameter zero. We also introduce a canonical independent increments process with "maximum skewness." We then, in section 3, represent "most" strictly stable processes as stochastic integrals with respect to this canonical process. This theorem in some sense includes and elucidates the known spectral representation theorem for symmetric stable processes ([2], [14], [9], [10]; see [8] for a discussion of this theorem).



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## Section 1. Definitions and Preliminary Notions

The stable distributions arise as the only possible limiting distributions of normalized sums of i.i.d. random variables. P. Levy discovered these distributions and also computed their characteristic functions. A random variable  $X$  is stable if whenever  $X_1$  and  $X_2$  are independent copies of  $X$ , and  $c_1$  and  $c_2$  are any positive constants, there exist real constants  $c > 0$  and  $d$  so that  $cX + d$  has the same distribution as  $c_1X_1 + c_2X_2$ . Equivalently, there should exist real constants  $\alpha$ ,  $\beta$ ,  $\mu$ , and  $v$  with  $0 < \alpha \leq 2$ ,  $-1 \leq \beta \leq 1$ , and  $v \geq 0$  so that  $X$  has characteristic function  $\phi_X(t) = E \exp(itX)$  of the form

$$(1.1) \quad -\log \phi_X(t) = \begin{cases} v|t|^\alpha (1 - i\beta \tan \frac{\alpha\pi}{2} \operatorname{sgn}(t)) - i\mu t & \text{if } \alpha \neq 1 \\ v|t| (1 + i\beta \frac{2}{\pi} \operatorname{sgn}(t) \log|t|) - i\mu t & \text{if } \alpha = 1 \end{cases}$$

Here  $\alpha$  is called the index of stability (in which case we call the distribution  $\alpha$ -stable),  $\beta$  is a skewness parameter,  $\mu$  is a location parameter, and  $v$  is a scale parameter. The skewness parameter  $\beta$  gives a measure of how much of the Lévy-Khintchine jump measure is placed on the positive and negative half lines. For example,  $\beta = +1$ ,  $-1$ , or  $0$  according as the measure is concentrated on the right half-line, the left half-line, or is symmetric. For a historically instructive and amusing account of these distributions and their characteristic functions, see P. Hall [7].

$X$  is called strictly stable if  $d$  above is identically zero; or equivalently, if  $\mu = 0$  in the case  $\alpha \neq 1$ , or if  $\beta = 0$  in the case  $\alpha = 1$ . If  $\alpha > 1$ , strictly  $\alpha$ -stable distributions have mean zero. Note that in the case  $\alpha \neq 1$ , all stable distributions are just translations of strictly stable distributions, but in the case  $\alpha = 1$ , the strictly stable distributions can have no skewness - they are just translations of symmetric Cauchy distributions. Because we

lose almost no generality if we consider only strictly stable distributions and processes in the case  $\alpha \neq 1$ , we will often do this. However for  $\alpha = 1$ , we must consider non-strictly stable distributions and processes to maintain generality.

If  $X$  is stable and satisfies (1.1) we will write  $X \sim S_\alpha(v, \beta, \mu)$ . Using (1.1), the reader may easily verify the following two facts.

Lemma 1.1 If  $X \sim S_\alpha(v, \beta, \mu)$  then for real  $c$ ,  $cX \sim S_\alpha(|c|^\alpha v, \text{sgn}(c)\beta, m)$  where  $m = c\mu$  if  $\alpha \neq 1$ , and  $m = c\mu - c \log|c| \frac{2}{\pi} v\beta$  if  $\alpha = 1$ .

Lemma 1.2 If  $X_1$  and  $X_2$  are independent with  $X_j \sim S_\alpha(v_j, \beta_j, \mu_j)$ , then for real  $c_1, c_2$ ,  $c_1X_1 + c_2X_2 \sim S_\alpha(v, \beta, \mu)$  where

$$(1.2) \quad v = |c_1|^\alpha v_1 + |c_2|^\alpha v_2,$$

$$(1.3) \quad \beta = (c_1^{<\alpha>} v_1 \beta_1 + c_2^{<\alpha>} v_2 \beta_2) / v, \quad \text{and}$$

$$(1.4) \quad \mu = \begin{cases} c_1 \mu_1 + c_2 \mu_2 & \text{if } \alpha \neq 1 \\ c_1 \mu_1 + c_2 \mu_2 - \frac{2}{\pi} (v_1 \beta_1 c_1 \log|c_1| + v_2 \beta_2 c_2 \log|c_2|) & \text{if } \alpha = 1 \end{cases}$$

Here, and in the sequel we use the convention that for complex  $z = re^{i\theta}$  and real  $p$ ,  $z^{<p>}$  denotes  $r^p e^{ip\theta}$ ; so that for  $x$  real,  $x^{<p>}$  denotes  $|x|^p \text{sgn}(x)$ .

We shall study stable processes in the remaining sections. But before embarking on that course, we give two propositions and an example, irrelevant to the rest of the paper, included only for whatever intrinsic interest they may have.

It is well-known that for  $0 < \rho < 1$  and  $0 < \alpha \leq 2$  that if  $A$  is a positive  $\rho$ -stable random variable, (i.e.  $A \sim S_\rho(v, 1, 0)$ ) and  $X$  is symmetric  $\alpha$ -stable and independent of  $A$ , that  $A^{1/\alpha} X$  is symmetric  $\rho\alpha$ -stable (see Feller [6], p. 596, ex. 9). Here is a slight generalization.

Proposition 1.3 Let  $A$  and  $X$  be independent with  $A \sim S_\rho(v_A, 1, 0)$  and  $X \sim S_\alpha(v_X, \beta, 0)$  where  $0 < \rho < 1$ ,  $0 < \alpha \leq 2$ , and  $\alpha \neq 1 \neq \rho\alpha$ . Then  $A^{1/2}X \sim S_{\rho\alpha}(\bar{v}, \bar{\beta}, 0)$ . In the case  $\alpha < 1$ , if  $\beta = -1, 0$ , or  $1$ , then  $\bar{\beta} = \beta$ .

Remark. The values for the parameters  $\bar{v}$  and  $\bar{\beta}$  are implicit in the proof, their explicit mention being avoided for aesthetic reasons.

Proof. WLOG assume that  $v_A = v_X = 1$ .

With some manipulation of (1.1) and application of some standard complex variable arguments, one can deduce that for complex  $w$  with  $\operatorname{Re} w \geq 0$ ,

$$(1.5) \quad E \exp(-wA) = \exp(-kw^\rho)$$

where  $k = 1/\cos \frac{\rho\pi}{2}$ , and we interpret  $w^\rho = e^{\rho \log w} = |w|^\rho e^{i\rho \arg(w)}$  with  $|\arg w| \leq \pi/2$ . Letting  $w > 0$  in (1.5) we can deduce that  $A > 0$  a.s. We compute the characteristic function of  $A^{1/\alpha}X$ . For  $\alpha \neq 1$ ,

$$\begin{aligned} E \exp(itA^{1/\alpha}X) &= EE(\exp(itA^{1/\alpha}X) | A) \\ &= E \exp\{-|tA^{1/\alpha}|^\alpha (1 - i\beta \tan \frac{\alpha\pi}{2} \operatorname{sgn}(t))\} \\ &= E \exp\{-[|t|^\alpha (1 - i\beta \tan \frac{\alpha\pi}{2} \operatorname{sgn}(t))] A\} \\ &= \exp\{-k[|t|^\alpha (1 - i\beta \tan \frac{\alpha\pi}{2} \operatorname{sgn}(t))]^\rho\} \\ &= \exp\{-k|t|^{\rho\alpha} (1 + \beta^2 \tan^2 \frac{\alpha\pi}{2}) [\cos \rho\theta - i \sin \rho\theta \operatorname{sgn}(t)]\} \\ &= \exp\{-k \cos \rho\theta (1 + \beta^2 \tan^2 \frac{\alpha\pi}{2}) |t|^{\rho\alpha} [1 - i \bar{\beta} \tan \frac{\rho\alpha\pi}{2} \operatorname{sgn}(t)]\} \end{aligned}$$

where  $\theta = \operatorname{Arctan}(\beta \tan \frac{\alpha\pi}{2})$  and  $\bar{\beta} = \tan \rho\theta / \tan(\rho\alpha\pi/2)$ . It is not difficult to see that  $|\bar{\beta}| \leq 1$  in all cases under consideration. Note also that if  $\alpha < 1$  and  $\beta = \pm 1$ , that  $\theta = \pm \frac{\alpha\pi}{2}$  and  $\bar{\beta} = \pm 1$  as well. However,  $\bar{\beta} \neq \beta$  in general.

Proposition 1.4 Let  $X \sim S_\alpha(v, \beta, 0)$  for  $\alpha \neq 1$ . Then for  $0 < p < \alpha$ ,  $E|X|^p$  exists and is given by

$$(1.6) \quad E|X|^p = v^{p/\alpha} c(p) f(p, \alpha, \beta)$$

where

$$c(p) = 2^{p-1} (p \int_0^\infty u^{-p-1} \sin^2 u \, du)^{-1}$$

and

$$f(p, \alpha, \beta) = \Gamma(1-p/\alpha) (1 + \beta^2 \tan^2 \frac{\alpha\pi}{2})^{p/2\alpha} \cos(\frac{p}{\alpha} \operatorname{Arctan}(\beta \tan \frac{\alpha\pi}{2}))$$

Further, for fixed  $v$ ,  $p$ , and  $\alpha$ ,  $E|X|^p$  is even in  $\beta$  and increasing in  $|\beta|$ .

Proof. First note that  $X$  is distributed as  $v^{1/\alpha} X_0$ , where  $X_0 \sim S_\alpha(1, \beta, 0)$ , and hence that  $E|X|^p = v^{p/\alpha} E|X_0|^p$ . An application of [16, Thm. 2] and a few changes of variable show that

$$E|X_0|^p = (4 \int_0^\infty u^{-p-1} \sin^2 u \, du)^{-1} 2^{p+1} \alpha^{-1} \int_0^\infty t^{-p/\alpha-1} (1 - e^{-t} \cos(st)) dt$$

where we have let  $s = \beta \tan \frac{\alpha\pi}{2}$ . We now compute the second integral.

$$\begin{aligned} \int_0^\infty t^{-p/\alpha-1} (1 - e^{-t} \cos(st)) dt &= (1 - e^{-t} \cos(st)) \left( -\frac{\alpha}{p} t^{-p/\alpha} \right) \Big|_0^\infty + \frac{\alpha}{p} \int_0^\infty t^{-p/\alpha} e^{-t} (\cos(st) + s \sin(st)) dt \\ &= 0 + \operatorname{Re} \frac{\alpha}{p} \int_0^\infty t^{-p/\alpha} (1 + is) e^{-t(1+is)} dt \\ &= \operatorname{Re} \frac{\alpha}{p} \int_0^\infty \left( \frac{z}{1+is} \right)^{-p/\alpha} e^{-z} dz \\ &= \operatorname{Re} \frac{\alpha}{p} (1+is)^{p/\alpha} \Gamma(1-p/\alpha) \\ &= \frac{\alpha}{p} \Gamma(1-p/\alpha) (1+s^2)^{p/2\alpha} \cos(\frac{p}{\alpha} \operatorname{Arctan} s) \end{aligned}$$

This shows (1.6). That  $E|X|^p$  is even in  $\beta$  is clear. To see that it is increasing in  $|\beta|$ , note first that  $\cos(\frac{p}{\alpha} \text{Arctan}(\beta \tan \frac{\alpha\pi}{2})) = \cos(\frac{p}{\alpha} \text{Arctan}(|\beta| \cdot |\tan \frac{\alpha\pi}{2}|))$ , and making the substitution  $\theta = \text{Arctan}(|\beta| |\tan \frac{\alpha\pi}{2}|)$  we see that  $\theta$  increases with  $|\beta|$  and  $0 \leq \theta < \frac{\pi}{2}$ . Now, letting

$$g(\theta) \equiv f(p, \alpha, \beta) / \Gamma(1-p/\alpha) = (1 + \tan^2 \theta)^{p/2\alpha} \cos(\frac{p}{\alpha} \theta)$$

$$= \cos(\frac{p}{\alpha} \theta) / (\cos \theta)^{p/\alpha} \equiv \cos(r\theta) / \cos^r \theta,$$

we can compute that  $g'(\theta) = r(\cos \theta)^{-r-1} \sin[\theta(1-r)] \geq 0$ . Hence  $f(p, \alpha, \beta)$  is increasing in  $|\beta|$ , as is  $E|X|^p$ . ■

Example 1.5 (Chen & Shepp [3]) Let  $Z_j$ ,  $j = 1, 2, 3$ , be independent, each with a  $S_1(1,1,0)$  distribution. Set  $X = Z_1 - Z_2$  and  $Y = Z_1 - Z_3$ . Clearly  $X$  and  $Y$  are symmetric about 0. Using Lemma 1.2, the reader may easily verify that  $X + Y$  is also symmetric, but not about 0!

## Section 2. Independent Increments Processes and Stochastic Integrals

Let  $I$  be an interval (finite or infinite) contained in  $\mathbb{R}$ . A process  $\{Z(s): s \in I\}$  is said to have independent increments if the random variables (increments)  $Z(t_1) - Z(s_1), \dots, Z(t_n) - Z(s_n)$  are independent whenever  $s_1 \leq t_1 \leq s_2 \leq t_2 \leq \dots \leq s_n \leq t_n$  and the  $s_j, t_j$  are in  $I$ . We characterize those processes with stable increments as follows.

Theorem 2.1 Fix  $0 < \alpha \leq 2$ . Let  $\{Z(s): s \in I\}$  be an independent increments process with each increment stable of index  $\alpha$ . Then there exist real-valued functions,  $\mu$ ,  $v$ , and  $\beta$  on  $I$  with  $v$  increasing,  $\beta$  measurable, and  $|\beta| \leq 1$ , such that each increment  $Z(t) - Z(s)$  has scale parameter  $|v(t) - v(s)|$ , skewness parameter  $\int_s^t \beta(x) dv(s) / |v(t) - v(s)|$ , and location parameter  $\mu(t) - \mu(s)$ . Conversely, any such choice of functions  $\mu$ ,  $v$ , and  $\beta$  defines such an independent increments process in the obvious way.

Remarks (i) If  $v(t) - v(s) = 0$ , the skewness parameter above is not defined, but is immaterial - we may take it to be zero.

(ii) This theorem has an obvious analog in the setting of independently scattered stable measures.

Proof. For any stable variable  $X$ , let us define  $V(X) = v$ ,  $B(X) = \beta$  and  $M(X) = \mu$ , where  $v, \beta, \mu$  are as in (1.1). Now pick  $t_0 \in I$  and set  $Z'(t) = Z(t) - Z(t_0)$  for all  $t \in I$ . Since each  $Z'(t)$  and each increment  $Z'(t) - Z'(s)$  is stable, we may define the functions  $v(t) = V(Z'(t)) \operatorname{sgn}(t - t_0)$ ,  $b(t) = B(Z'(t))$ ,  $w(t) = b(t)v(t)$ , and  $\mu(t) = M(Z'(t))$ .

We claim that for  $s \leq t$ ,  $v(t) - v(s) = V(Z(t) - Z(s))$ , and thus  $v$  is increasing and  $Z(t) - Z(s)$  has scale parameter  $|v(t) - v(s)|$  for arbitrary  $s, t$ . We verify the claim only in the case  $s \leq t \leq t_0$ , leaving the similarly proved cases  $s \leq t_0 \leq t$  and  $t_0 \leq s \leq t$  to the reader. In the case at hand,

$$\begin{aligned}
v(t) - v(s) &= V(Z(t)-Z(t_0))\text{sgn}(t-t_0) - V(Z(s)-Z(t_0))\text{sgn}(s-t_0) \\
&= V(Z(s)-Z(t_0)) - V(Z(t)-Z(t_0)) \\
&= V(Z(t_0) - Z(t) + Z(t) - Z(s)) - V(Z(t_0)-Z(t)) \\
&= [V(Z(t_0)-Z(t))+V(Z(t)-Z(s))] - V(Z(t_0)-Z(t)) \text{ by (1.2)} \\
&= V(Z(t)-Z(s))
\end{aligned}$$

Since  $v$  is increasing, it is measurable, and induces a Lebesgue-Stieltjes measure  $dv$  on  $I$ .

Now note that by Lemma 1.2, if  $X_1$  and  $X_2$  are independent and stable, that

$$B(X_1+X_2)V(X_1+X_2) = B(X_1)V(X_1) + B(X_2)V(X_2)$$

Hence for  $t_0 \leq s \leq t$ , letting  $X_1 = Z(t) - Z(s)$  and  $X_2 = Z(s) - Z(t_0)$ ,

$$\begin{aligned}
w(t) - w(s) &= b(t)v(t) - b(s)v(s) \\
&= B(X_1+X_2)V(X_1+X_2) - B(X_2)V(X_2) \\
&= B(X_1)V(X_1) \\
&= B(Z(t)-Z(s))[v(t)-v(s)]
\end{aligned}$$

It is easy to verify in like fashion that the equation

$$w(t) - w(s) = B(Z(t)-Z(s))[v(t)-v(s)]$$

holds true in the cases  $s \leq t_0 \leq t$  and  $s \leq t \leq t_0$ . Hence

$$(2.1) \quad w(t) - w(s) = B(Z(t)-Z(s))|v(t)-v(s)|$$

for arbitrary  $s, t$ .

In any case,  $|w(t)-w(s)| \leq v(t)-v(s)$  for  $s \leq t$ . Hence for  $a \leq t_0 \leq t_1 \leq \dots \leq t_n \leq b$ ,  $\sum_{j=1}^n |w(t_j)-w(t_{j-1})| \leq \sum_{j=1}^n v(t_j) - v(t_{j-1}) \leq v(b) - v(a)$ .

This shows that  $w$  is of bounded variation on any finite interval contained in  $I$  and induces a Lebesgue - Stieltjes measure  $dw$  on  $I$  which is absolutely continuous with respect to  $dv$ . We may set  $\beta(x) = \frac{dw}{dv}(x)$ . Clearly  $|\beta(x)| \leq 1$

a.e.  $[dv]$  and by (2.1),  $\int_s^t \beta(x) dv(x) / |v(t) - v(s)| = B(Z(t) - Z(s))$ , which is the skewness parameter of  $Z(t) - Z(s)$ .

Finally, it is easy to check (using Lemma 1.2) that  $M(Z(t) - Z(s)) = \mu(t) - \mu(s)$  for all  $s, t$ . This holds even in the case  $\alpha = 1$ , since  $\log|\pm 1| = 0$ . ■

We now introduce a particular independent increments process of some importance to us in Section 3. Fix  $\alpha$  and let  $\psi_\alpha(t)$  be the right-hand side of (1.1) with  $v = \beta = 1$  and  $\mu = 0$ . Since  $\exp(-\psi_\alpha(t))$  is the characteristic function of an infinitely divisible distribution, there is an independent increments process  $\{R(s) : s \in \mathbb{R}\}$  with characteristic function

$$E \exp(itR(s)) = \exp(-|s|\psi_\alpha(t))$$

(see, e.g. [1]). This process  $R$  will be called the *canonical totally right-skewed  $\alpha$ -stable Levy process*. The corresponding function parameters of Theorem 2.1 for  $R$  are  $v(t) = t$ ,  $\beta(t) \equiv 0$ . For  $0 < \alpha < 1$ ,  $R$  is an increasing process.

We now wish to produce a Wiener-type stochastic integral with respect a process with independent stable increments. The development here is standard, but some interesting things happen in the case  $\alpha = 1$ .

To lay the groundwork for the integral in this case we introduce the Orlicz space  $L \log^+ L$ , defined as follows. Let  $(M, d\mu)$  be a finite measure space and let  $\phi(x) = x \log^+ x$  for  $x \geq 0$ . Note that  $\phi$  is positive, convex, and increasing on  $[0, \infty)$ , and that  $\phi$  satisfies the so-called  $\Delta_2$  condition at  $\infty$ , i.e.  $\lim_{x \rightarrow \infty} \phi(2x)/\phi(x) < \infty$ . This is enough to guarantee that the Orlicz space based on  $\phi$ , which we call  $L \log^+ L(M, d\mu)$ , defined by

$$L \log^+ L = \{\text{measurable } f: M \rightarrow \mathbb{R} : \int_M \phi(|f|) d\mu < \infty\},$$

is a Banach space under the norm

$$\|f\|_{L\log^+L} = \inf\{c>0: \int_M \phi(|f|/c) d\mu < 1\}$$

and that, in this space, simple functions form a dense subset.

What is important for us is that convergence of, say,  $f_n$  to  $f$  in  $L\log^+L$  implies the same convergence in  $L^1$ , and that of the integrals  $\int (f_n - f) \log |f_n - f|$  to zero, as is shown in the following

Lemma 2.2 For  $\|f\|_{L\log^+L} = c \leq e^{-2}$

$$(2.2) \quad \|f\|_1 \leq K_1 c \quad \text{and}$$

$$(2.3) \quad \int_M |f \log |f|| d\mu \leq K_2 \sqrt{c} |\log c|$$

the constants  $K_1$  and  $K_2$  depending only on  $\mu(M)$ .

Proof Note that by the monotone convergence theorem, we have that

$$\int \frac{|f|}{c} \log^+ \left( \frac{|f|}{c} \right) \leq 1. \quad \text{Hence}$$

$$1 \geq \int_{\{|f|>c\}} \frac{|f|}{c} \log \frac{|f|}{c} \geq \int_{\{|f|>ce\}} \frac{|f|}{c}$$

so that 
$$\int_{\{|f|>ce\}} |f| \leq c.$$

Since 
$$\int_{\{|f|\leq ce\}} |f| \leq ce\mu(M),$$
 (2.2) is proved with  $K_1 = 1 + e\mu(M)$ .

For (2.3), examination of  $|x \log x|$  for  $x \in [0,1]$  shows that for  $\delta \leq e^{-1}$ ,

$$(2.4) \quad \begin{aligned} \int |f \log |f|| &= \int_{\{|f|>1\}} + \int_{\{\delta \leq |f| \leq 1\}} + \int_{\{|f|<\delta\}} \\ &\leq \int_{\{|f|>1\}} |f| \log |f| + \mu\{|f|>\delta\} e^{-1} + \delta |\log \delta| \mu(M) \end{aligned}$$

Also,

$$1 \geq \int \frac{|f|}{c} \log^+ \frac{|f|}{c} \geq \int_{\{|f|>1\}} \frac{|f|}{c} \log |f|$$

which shows that  $\int |f| \log |f| \leq c$ . Noting that  $\mu\{|f|>\delta\} \leq \|f\|_1 \delta^{-1}$  by the Markov inequality, and choosing  $\delta = \sqrt{c}$  gives us through (2.2) and (2.4) that

$$\begin{aligned} \int |f \log |f|| &\leq c + \|f\|_1 \delta^{-1} e^{-1} + \frac{1}{2} \mu(M) \sqrt{c} |\log c| \\ &\leq c + \sqrt{c} (e^{-1} + \mu(M)) + \frac{1}{2} \mu(M) \sqrt{c} |\log c|, \end{aligned}$$

establishing (2.3). ■

Now, let  $\{Z(s): s \in I\}$  be a process with  $\alpha$ -stable independent increments as in Theorem 2.1. For ease and simplicity we will let  $Z$  be such that  $\mu(t) \equiv 0$ , although processes with sufficiently well-behaved  $\mu(t)$  can be handled in a straightforward way. Also, we assume in the case  $\alpha = 1$  that  $v(I) < \infty$ .

For a step function  $f$  of the form  $f = \sum_{j=1}^n c_j 1_{(s_{j-1}, s_j]}]$  (where  $c_j \in \mathbb{R}$ ,  $s_j \in I$ ,  $s_{j-1} < s_j$  for all  $j$ ) we define

$$(2.5) \quad S(f) = \int_I f(s) dZ(s) = \sum_{j=1}^n c_j [Z(s_j) - Z(s_{j-1})]$$

**Theorem 2.3** The stochastic integral  $S(f) = \int_I f(s) dZ(s)$  can be defined for all  $f$  in the space  $L^\alpha(I, dv)$  (resp.  $L \log^+ L(I, dv)$ ) in the case  $\alpha \neq 1$  (resp.  $\alpha = 1$ ) so that it is linear, agrees with (2.5) for step functions, and satisfies  $S(f) \sim S_\alpha(v_f, \beta_f, \mu_f)$  where

$$v_f = \int_I |f(x)|^\alpha dv(x) = \|f\|_\alpha^\alpha,$$

$$\beta_f = v_f^{-1} \int_I [f(x)]^{<\alpha>} \beta(x) dv(x), \quad \text{and}$$

$$\mu_f = \begin{cases} 0 & \alpha \neq 1 \\ -\frac{2}{\pi} \int_I f(x) \log |f(x)| \beta(x) dv(x) & \alpha = 1 \end{cases}$$

Further  $S(f)$  and  $S(g)$  are independent if and only if  $fg = 0$  a.e.  $[dv]$ .

Proof Note that by Lemma 1.2 and Theorem 2.1, for a step function  $f$  as in (2.5) we have  $S(f) - S_\alpha(v_f, \beta_f, \mu_f)$  where

$$v_f = \sum_{j=1}^n |c_j|^\alpha [v(t_j) - v(t_{j-1})]$$

$$\begin{aligned} \beta_f &= v_f^{-1} \sum_{j=1}^n c_j^{<\alpha>} [v(t_j) - v(t_{j-1})] \int_{t_{j-1}}^{t_j} \beta(x) dv(x) / [v(t_j) - v(t_{j-1})] \\ &= v_f^{-1} \int f^{<\alpha>} \beta dv \end{aligned}$$

and

$$\mu_f = \begin{cases} 0 & \alpha \neq 1 \\ \frac{2}{\pi} \sum_{j=1}^n c_j \log |c_j| [v(t_1) - v(t_{j-1})] \int_{t_{j-1}}^{t_j} \beta(x) dv(x) / [v(t_1) - v(t_{j-1})] & \alpha = 1 \end{cases}$$

$$= \begin{cases} 0 & \alpha \neq 1 \\ -\frac{2}{\pi} \int_I f \log |f| \beta dv & \alpha = 1 \end{cases}$$

Thus the claim concerning the distribution of  $S(f)$  is true for step functions.

Also, for such functions, the claim of linearity is easily seen to be true.

To define  $S(f)$  for general  $f$  in  $L^\alpha$  (resp  $L \log^+ L$ ), pick a sequence of step functions  $f_n$  converging in norm (quasi-norm, if  $\alpha < 1$ ) to  $f$ . Note that

$\int f_n dZ - \int f_m dZ = S(f_n - f_m)$  by linearity on step functions, and so

$\int f_n dZ - \int f_m dZ$  has scale parameter  $v_{f_n - f_m} = \|f_n - f_m\|_\alpha^\alpha$  and location parameter  $\mu_{f_n - f_m} = 0$  if  $\alpha \neq 1$  and  $\mu_{f_n - f_m} = -\frac{2}{\pi} \int (f_n - f_m) \log |f_n - f_m| \beta dv$  if  $\alpha = 1$ . Since these

parameters approach zero as  $n, m$  become large (cf. Lemma 2.2 in the case  $\alpha=1$ ), we conclude that  $\{\int f_n dZ\}$  is a Cauchy sequence in the metric of convergence in probability. We are then free to define  $\int f dZ = \lim_{n \rightarrow \infty} \text{prob} \int f_n dZ$ .

Since  $\|f_n\|_\alpha \rightarrow \|f\|_\alpha$ ,  $\int f_n^{<\alpha>} \beta dv \rightarrow \int f^{<\alpha>} \beta dv$ , and  $\int f_n \log |f_n| \beta dv \rightarrow \int f \log |f| \beta dv$ , Levy's continuity theorem guarantees that  $v_f$ ,  $\beta_f$ , and  $\mu_f$  are as advertised.  $S(\cdot)$  is linear by continuity.

To verify the claim concerning independence, we observe that  $S(f)$  and  $S(g)$  are independent if and only if

$$\phi_{S(f), S(g)}(s, t) = \phi_{S(f)}(s) \cdot \phi_{S(g)}(t)$$

for all  $s$  and  $t$ . If  $fg = 0$  a.e.  $[dv]$ , one can verify easily, using the form of the characteristic function for these stochastic integrals, that this equation holds. Conversely, for  $S(f)$  to be independent of  $S(g)$ , we must have that

$$|\phi_{S(f), S(g)}(s, t)| = |\phi_{S(f)}(s)| |\phi_{S(g)}(t)|$$

which implies that

$$\|sf + tg\|_\alpha^\alpha = \|sf\|_\alpha^\alpha + \|tg\|_\alpha^\alpha$$

and hence that

$$\|f+g\|_{\alpha}^{\alpha} + \|f-g\|_{\alpha}^{\alpha} = 2(\|f\|_{\alpha}^{\alpha} + \|g\|_{\alpha}^{\alpha}) \quad .$$

That this implies  $fg = 0$  is given by the result of Lamperti [11, Corollary 2.1]. ■

### Section 3. Stable Processes and their Spectral Representations

A stochastic process  $\{X_t: t \in T\}$  will be called a *stable process* if each random vector  $\bar{X} = (X_{t_1}, \dots, X_{t_n})$ ,  $t_j \in T$ , is multivariate stable, i.e. if for independent copies  $\bar{X}_1$  and  $\bar{X}_2$  of  $\bar{X}$  and positive constants  $c_1$  and  $c_2$ , there exist constants  $c > 0$  and  $\bar{d} \in \mathbb{R}^n$  so that  $c_1 \bar{X}_1 + c_2 \bar{X}_2$  is distributed as  $c\bar{X} + \bar{d}$ . As before,  $\{X_t\}$  is a *strictly stable process* if  $\bar{d}$  is zero for all  $t_j$  and  $c_j$ . Clearly if  $\bar{X}$  is stable, then  $\bar{\lambda} \cdot \bar{X}$  is stable for any  $\bar{\lambda} \in \mathbb{R}^n$ . The converse is not true in general (see [13]), but is true in the strictly stable case (see [4]).

Levy [12] and Feldheim [5] have shown that a random vector  $\bar{X}$  in  $\mathbb{R}^n$  is multivariate stable if and only if there exists  $\alpha \in (0, 2]$ ,  $\bar{\mu} \in \mathbb{R}^n$ , and a finite measure  $\Gamma$  on the unit sphere  $S^n$  of  $\mathbb{R}^n$  such that the characteristic function  $\phi_{\bar{X}}$  of  $\bar{X}$  satisfies

$$(3.1) \quad -\log \phi_{\bar{X}}(\bar{t}) = \begin{cases} \int_{S^n} (|\bar{t} \cdot \bar{s}|)^\alpha - i \tan \frac{\alpha\pi}{2} (\bar{t} \cdot \bar{s})^{\langle \alpha \rangle} d\Gamma(\bar{s}) - i\bar{t} \cdot \bar{\mu} & \text{if } \alpha \neq 1 \\ \int_{S^n} (|\bar{t} \cdot \bar{s}| + i \frac{2}{\pi} (\bar{t} \cdot \bar{s}) \log |\bar{t} \cdot \bar{s}|) d\Gamma(\bar{s}) - i\bar{t} \cdot \bar{\mu} & \text{if } \alpha = 1 \end{cases}$$

When  $n = 1$ , this agrees with (1.1) with  $v = \Gamma(S^1)$  and  $\bar{s} = [\Gamma(1) - \Gamma(-1)]/\Gamma(S^1)$ . Also observe that this shows that the random variable  $\bar{t} \cdot \bar{X}$  has a  $S_\alpha(v, \bar{s}, \bar{\mu})$  distribution, where  $v = \int |\bar{t} \cdot \bar{s}|^\alpha d\Gamma(\bar{s})$ ,  $\bar{s} = v^{-1} \int (\bar{t} \cdot \bar{s})^{\langle \alpha \rangle} d\Gamma(\bar{s})$ , and  $\bar{\mu} = \bar{t} \cdot \bar{\mu}$  in case  $\alpha \neq 1$  and  $\bar{\mu} = \bar{t} \cdot \bar{\mu} - \frac{2}{\pi} \int (\bar{t} \cdot \bar{s}) \log |\bar{t} \cdot \bar{s}| d\Gamma(\bar{s})$  if  $\alpha = 1$ .

We warn the reader that in case  $\alpha = 1$  this shows  $\bar{\mu}$  of (3.1) is not the vector of location parameters of the individual  $X_j$ 's, but rather  $X_j$  has location parameter  $\mu_j = \frac{2}{\pi} \int s_j \log |s_j| d\Gamma(\bar{s})$ .

In the representation (3.1), strict stability is equivalent to  $\bar{\mu} = 0$  if  $\alpha \neq 1$ , and to  $\int (\bar{t} \cdot \bar{s}) d\bar{r}(\bar{s}) = 0$  for all  $\bar{t}$  if  $\alpha = 1$ . The latter occurs if and only if we may choose  $\Gamma$  symmetric. Also,  $\bar{X}$  is symmetric about 0 (for any  $\alpha < 2$ ) if and only if  $\bar{\mu} = 0$  and we may choose  $\Gamma$  symmetric.

Now, for an arbitrary index set  $T$ , pick arbitrary functions  $\{f_t: t \in T\}$  in  $L^\alpha([0,1], \lambda)$  if  $\alpha \neq 1$  (resp.  $L \log^+ L([0,1], \lambda)$  if  $\alpha = 1$ ) and set

$$X_t = \int_{[0,1]} f_t(s) dR(s)$$

where  $R$  is the canonical totally right-skewed  $\alpha$ -stable Lévy process (see section 2). We now argue that  $\{X_t\}$  is an  $\alpha$ -stable process (strict, if  $\alpha \neq 1$ ). Let  $\tilde{R}$  be the canonical right-skewed  $\alpha$ -stable Lévy process on  $[0,2]$ . Let

$$\begin{aligned} f_{t,1}(x) &= \begin{cases} f_t(x) & x \in [0,1] \\ 0 & x \in (1,2] \end{cases} \quad \text{and} \\ f_{t,2}(x) &= \begin{cases} 0 & x \in [0,1] \\ f_t(x-1) & x \in (1,2]. \end{cases} \end{aligned}$$

Clearly the processes  $\{X_{t,1}\}$  and  $\{X_{t,2}\}$  defined by  $X_{t,1} \equiv \int f_{t,1} d\tilde{R}$  and  $X_{t,2} \equiv \int f_{t,2} d\tilde{R}$  are independent and distributed as  $\{X_t\}$  by Theorem 2.3. Straightforward (and arduous, if  $\alpha=1$ ) computations with the joint characteristic function and formulae from Theorem 2.3 show for  $c_1, c_2 > 0$  that  $c_1(X_{t_1,1}, \dots, X_{t_n,1}) + c_2(X_{t_1,2}, \dots, X_{t_n,2})$  is distributed as  $(c_1^\alpha + c_2^\alpha)^{1/\alpha} (X_{t_1}, \dots, X_{t_n}) + \bar{d}$ , where  $\bar{d} = 0$  when  $\alpha \neq 1$ , and  $\bar{d} = k(\int f_{t_1} d\lambda, \dots, \int f_{t_n} d\lambda)$  with  $k = \frac{2}{\pi}[(c_1 + c_2) \log(c_1 + c_2) - c_1 \log c_1 - c_2 \log c_2]$  when  $\alpha=1$ . Hence,  $\{X_t\}$  is in fact a stable process. It is always strictly stable if  $\alpha \neq 1$  and is strictly stable in the case  $\alpha=1$  if and only if

$$\int f_t d\lambda = 0 \quad \text{for all } t.$$

It is the purpose of this section to show that practically all stable processes can be represented in this way, i.e. as a set of stochastic integrals of appropriate functions against this canonical totally right-skewed  $\alpha$ -stable Lévy process. Our method is in essence the same as that of Kanter in the case of symmetric stable processes (see [9] in view of [14]). That is, we shall use the finite-dimensional representation (3.1) in conjunction with an imbedding theorem (Theorem 3.1) similar to the main theorem of [9]. We comment that another route to this representation is to use the Hilbert space analogue of equation (3.1) due to Keulbs [10] and to proceed as he has in his last section, carrying the skewness parameters along. We prefer to proceed in the manner described because we feel it is more basic and direct, and allows us to present Theorem 3.1.

Theorem 3.1 (i) Let  $0 < p < \infty$ . Let  $L$  be a (real or complex) separable Fréchet space such that for each finite-dimensional subspace  $M$  there is a linear isometry  $J_M: M \rightarrow L^p[0,1]$ . Then all of  $L$  is linearly isometrically imbeddable in  $L^p$  via  $J: L \rightarrow L^p[0,1]$ .

(ii) If in addition we assume that for fixed  $f$ ,  $\int (J_M f)^{<p>} d\lambda$  is independent of  $M$  containing  $f$ , we can then choose  $J$  above so that

$$(3.2) \quad \int (Jf)^{<p>} d\lambda = \int (J_M f)^{<p>} d\lambda$$

for all  $f$  in  $L$  and all  $M$  containing  $f$ .

Remarks Part (i) of the theorem in the real case is due to Bretagnolle et al [2] in the case  $p \geq 1$  and to Schreiber [15] in the case  $0 < p < 1$ .

Both these papers prove this without the separability hypothesis on  $L$  and with the range of  $J$  being  $L^p(E, \mu)$  with  $(E, \mu)$  a suitable measure space. Kanter [9] has given a shorter proof in the real separable case for  $0 < p \leq 2$ . Part (ii) is our main concern here - part (i) is included because it follows easily from our proof (by ignoring the extra assumption in (ii) and its consequences), and because our proof is a still simpler and more general version of Kanter's.

Proof Since  $L$  is separable, we may find  $f_1, f_2, \dots$  in  $L$  such that  $\overline{\text{sp}\{f_1, f_2, \dots\}} = L$  and  $\sum_n \|f_n\| \leq 1$  (where  $\|\cdot\|$  is the quasi-norm on  $L$ ). Define  $M_n = \text{sp}\{f_1, \dots, f_n\}$  and let  $J_n$  be the hypothesized  $J_{M_n}$ .

Setting  $r = p^{-1}$  we have that for  $n \geq m$  and scalars  $\lambda_j$ ,

$$(3.3) \quad \left\| \sum_{j=1}^m \lambda_j J_n f_j \right\|_p^p = \left\| \sum_{j=1}^m \lambda_j f_j \right\|^r$$

$$= \left\| \sum_{j=1}^m \lambda_j J_m f_j \right\|_p^p,$$

and also (under the assumption of (ii))

$$(3.4) \quad \int \left( \sum_{j=1}^m \lambda_j J_n f_j \right)^{<p>} d\lambda = \int \left( \sum_{j=1}^m \lambda_j J_m f_j \right)^{<p>} d\lambda.$$

Call  $F_n = \left( \sum_{j=1}^n |J_n f_j|^p \right)^{1/p}$  and define  $d\mu_n = F_n^p d\lambda$ . By (3.3) and (3.4),

$$(3.5) \quad \int \left| \sum_{j=1}^m \lambda_j J_n f_j / F_n \right|^p d\mu_n = \int \left| \sum_{j=1}^m \lambda_j J_m f_j / F_m \right|^p d\mu_m$$

and

$$(3.6) \quad \int \left( \sum_{j=1}^m \lambda_j J_n f_j / F_n \right)^{<p>} d\mu_n = \int \left( \sum_{j=1}^m \lambda_j J_m f_j / F_m \right)^{<p>} d\mu_m$$

Now call  $B = \{(z_1, z_2, \dots) \in K^\infty : \sum |z_j|^p \leq 1\}$  where  $K$  is the scalar field  $\mathbb{R}$  or  $\mathbb{C}$ , and give  $B$  the relative topology as a subspace of the product space

$D \equiv \prod_{j=1}^{\infty} \{z \in K : |z_j| \leq 1\}$ . Since  $B$  is closed in  $D$ , and  $D$  is compact (by the Tychonoff theorem), we conclude  $B$  is compact.

Define measures  $\nu_n$  on  $B$  by

$$\nu_n(E) = \mu_n\{(J_n f_1/F_n, \dots, J_n f_n/F_n, 0, 0, \dots) \in E\}$$

for Borel sets  $E$  of  $B$ . With this notation, equations (3.5) and (3.6) now read

$$(3.7) \quad \int_B \left| \sum_{j=1}^m \lambda_j z_j \right|^p d\nu_n(z) = \int_B \left| \sum_{j=1}^m \lambda_j z_j \right|^p d\nu_m(z)$$

and

$$(3.8) \quad \int_B \left( \sum_{j=1}^m \lambda_j z_j \right)^{<p>} d\nu_n(z) = \int_B \left( \sum_{j=1}^m \lambda_j z_j \right)^{<p>} d\nu_m(z)$$

Note that  $\nu_n(B) = \mu_n([0,1]) = \int_0^1 F_n^p d\lambda = \sum_{j=1}^n \|J_n f_j\|_p^p = \sum_{j=1}^n \|f_j\|^r \leq 1$ . Since

$B$  is compact, the Banach-Alaoglu theorem guarantees the existence of a subsequence  $\{\nu_{n_k}\}$  and a finite measure  $\nu$  on  $B$  with  $\{\nu_{n_k}\}$  converging weakly to  $\nu$ . Since the functions  $z \rightarrow \left| \sum_{j=1}^m \lambda_j z_j \right|^p$  and  $z \rightarrow \left( \sum_{j=1}^m \lambda_j z_j \right)^{<p>}$  are continuous on  $B$  for each  $m$  and  $\lambda_1, \dots, \lambda_m \in K$ , it follows from (3.7) and (3.8) that

$$(3.9) \quad \int_B \left| \sum_{j=1}^m \lambda_j z_j \right|^p d\nu(z) = \left\| \sum_{j=1}^m \lambda_j f_j \right\|^r$$

and

$$(3.10) \quad \int_B \left( \sum_{j=1}^m \lambda_j z_j \right)^{<p>} d\nu(z) = \int_M \left( \sum_{j=1}^m \lambda_j f_j \right)^{<p>} d\lambda$$

for any  $M$  containing  $M_m$ .

Now define  $U: \text{sp}\{f_1, f_2, \dots\} \rightarrow L^P(B, d\nu)$  to be the linear extension of the map  $f_n \rightarrow p_n$  where  $p_n(z) = z_n$ . By (3.9),  $U$  is an isometry and hence has a unique extension (also called  $U$ ) by continuity to all of  $L$ . Also, under the assumption of (ii), this extension must have the property that

$$(3.11) \quad \int (Uf)^{<P>} d\nu = \int (J_M f)^{<P>} d\lambda$$

for all  $f$  and finite-dimensional  $M$  containing  $f$ , by (3.10). By using a measure algebra isomorphism of  $B$  into  $[0,1]$ , we may find an isometry  $V$  of  $L^P(B, d\nu)$  into  $L^P([0,1], \lambda)$  which satisfies  $\int (Vf)^{<P>} d\lambda = \int f^{<P>} d\nu$ . Setting  $J = VU$  completes the proof. ■

With this theorem in hand we can prove the representation theorem previously advertised.

Let  $\{X_t: t \in T\}$  be an arbitrary stochastic process. We say that it satisfies condition  $S$  if there exists a countable set  $T' \subseteq T$  such that every  $X_t$  is a limit in probability of a sequence from the set of all finite linear combinations  $\sum \lambda_j X_{t_j}$ ;  $\lambda_j \in \mathbb{R}$ ,  $t_j \in T'$  (see [8] and [10]). Practically all processes of interest will satisfy this condition, including those continuous in probability and those defined on a countable index set  $T$ .

Let  $L_X$  be the linear span of the  $X_t$ 's, closed under the metric of convergence in probability. It is not a difficult exercise to show that condition  $S$  is equivalent to the requirement that  $L_X$  be separable.

**Theorem 3.2** Let  $\alpha \neq 1$ .  $\{X_t: t \in T\}$  is a strictly stable process of index  $\alpha$  satisfying condition  $S$  if and only if there exists a collection of functions  $\{f_t: t \in T\} \subseteq L^\alpha[0,1]$  such that the process  $\{Y_t\}$  defined by  $Y_t = \int f_t dR$  is distributionally equivalent to  $\{X_t\}$ .

Proof For the "if" part of the theorem, we note that  $\{Y_t\}$  above is a strictly  $\alpha$ -stable process by arguments given at the beginning of this section. To see that  $\{Y_t\}$  (and hence  $\{X_t\}$ ) satisfies condition S, note that  $L^\alpha[0,1]$  (and hence any subspace thereof) is separable, and the closed linear extension of the map  $f_t \rightarrow Y_t$  is an isomorphism of  $\overline{\text{sp}\{f_t\}}_L^\alpha$  onto  $L_Y$ .

We now prove the "only if" part. Clearly, every element of  $L_X$  is strictly stable of index  $\alpha$ , and moreover,  $\{Z: Z \in L_X\}$  is a strictly  $\alpha$ -stable process. Now let  $M$  be a finite-dimensional subspace of  $L_X$ , and let  $Z_1, \dots, Z_n$  be such that  $M = \text{sp}\{Z_1, \dots, Z_n\}$ . Since  $\bar{Z} \equiv (Z_1, \dots, Z_n)$  is jointly strictly  $\alpha$ -stable, we may write by (3.1).

$$(3.12) \quad -\log \bar{Z}(\bar{t}) = \int_{S^n} (|\bar{t} \cdot \bar{s}|^\alpha - i \tan \frac{\alpha\pi}{2} (\bar{t} \cdot \bar{s})^{<\alpha>}) d\Gamma(\bar{s}).$$

for some finite measure  $\Gamma$  on  $S^n$ .

Define  $g_j: S^n \rightarrow \mathbb{R}$  by  $g_j(\bar{s}) = s_j(\Gamma(S^n))^{-1/\alpha}$ . Now pick a measure-preserving set transformation  $T$  of  $(F, (\Gamma(S^n))^{-1} d\Gamma)$  into  $(B, d\lambda)$ . Here,

$F = \sigma\{g_j: j = 1, \dots, n\}$ ,  $B = \{\text{Borel sets of } [0,1]\}$ ,

and  $d\lambda$  is Lebesgue measure. Set  $h_j = Tg_j$ . It is evident from (3.12) that

$$(3.13) \quad \begin{aligned} -\log \bar{Z}(\bar{t}) &= \int_{S^n} (|\sum t_j g_j(\bar{s})|^\alpha - i \tan \frac{\alpha\pi}{2} (\sum t_j g_j(\bar{s}))^{<\alpha>} (\Gamma(S^n))^{-1} d\Gamma(\bar{s})) \\ &= \int_0^1 |\sum t_j h_j(x)|^\alpha d\lambda(x) - i \tan \frac{\alpha\pi}{2} \int_0^1 (\sum t_j h_j(x))^{<\alpha>} d\lambda(x) \end{aligned}$$

Now we metrize  $L_X$  as follows. For any  $Y \in L_X$  we set  $\|Y\| = (-\log |E e^{iY}|)^{1/\alpha}$ . By (3.13),  $\|\cdot\|$  a quasi-norm (a true norm, if  $\alpha > 1$ ) on  $L_X$  which metrizes convergence in probability. Also, for  $L_X$  so normed, the linear extension of the map  $Z_j \rightarrow h_j$ , call it  $J_M$ , is an isometry of  $M$  into  $L^\alpha[0,1]$ . We also see that for each  $Z$  in  $M$ ,  $\int_0^1 (J_M(Z))^{<\alpha>} d\lambda$  is just the skewness parameter of  $Z$ ,

multiplied by  $\|Z\|^{\alpha-1}$ , and hence does not depend on  $M$  containing  $f$ .

We can now apply Theorem 3.1 (ii) to conclude that there is an isometry  $J: L_X \rightarrow L^\alpha[0,1]$  with  $\int (J(Z))^{\alpha} d\lambda$  being the skewness parameter of  $Z$ , multiplied by  $\|Z\|^{\alpha-1}$ . Setting  $f_t = J(X_t)$  and recalling Theorem 2.3 completes the proof. ■

The author has not as of yet been able to obtain an analogous version of this theorem in the case  $\alpha = 1$ . One difficulty stems in part from the curious behavior of the location parameter (cf. Lemma 1.1, (1.4), example 1.5, Theorem 2.3, and the comments following (3.1)). It is not a linear function of the variables in the process as is the case if  $\alpha \neq 1$ . Another difficulty comes from the fact that with respect to scale and skewness, the appropriate "imbedding space" is  $L^1$ ; yet for the location parameter to exist, integrals of the form  $\int f \log|f| d\lambda$  must be defined in some appropriate sense. Hence it is not clear what the appropriate space of integrable functions should be -  $L^1$  is too large (but has the "right" norm in one sense), and  $L \log^+ L$  may be too small or may have an altogether inappropriate norm.

We close by making a few comments regarding Theorem 3.2. First, we observe we have lost little generality in assuming strict stability, since subtracting location parameters (which are means if  $\alpha > 1$ ) from a general stable process yields a strictly stable process in the case  $\alpha \neq 1$ . Secondly, we mention that in some sense the skewness of elements of the process are mirrored in the skewness of the corresponding representing functions. More specifically, observe that a process  $X_t \equiv \int f_t dR$  has all variables totally right-skewed (resp. left-skewed), i.e., all variables have skewness parameter  $+1$  (resp.  $-1$ ), if and only if  $f_t \geq 0$  (resp.  $f_t \leq 0$ ) a.e.  $[d\lambda]$  for each  $t$ . Also if  $\{X_t\}$  is a symmetric stable process, setting

$$\tilde{f}_t(x) = \begin{cases} f_t(2x) & \text{for } 0 \leq x \leq \frac{1}{2} \\ -f_t(2x-1) & \text{for } \frac{1}{2} < x \leq 1 \end{cases}$$

and observing that  $\{\int \tilde{f}_t dR\}$  is distributed as  $\{X_t\}$  gives us that  $\{X_t\}$  is a symmetric stable process if and only if it has a representation  $\{\int \tilde{f}_t dR\}$  with each  $\tilde{f}_t$  (and each linear combination  $\sum \lambda_j \tilde{f}_{t_j}$ ) symmetrically distributed. Also, our theorem relates to the spectral representation theorem for symmetric stable processes (see [8]) as follows: if the symmetric  $\alpha$ -stable process  $\{X_t\}$  is represented by  $\{\int f_t dZ\}$ , where  $Z$  is a symmetric  $\alpha$ -stable Lévy process, then defining  $\tilde{f}_t$  from  $f_t$  as above, the process  $\{\int \tilde{f}_t dR\}$  also represents  $\{X_t\}$ .

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**END**

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